

Consumer Harm from the Waterbed Effect: A Comment on Inderst and Valletti (2011)

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Inderst and Valletti (2011, henceforth IV) argue that a merger of downstream buyers can locally harm consumers through the “waterbed effect.” Their Proposition 6 gives a sufficient condition for consumer surplus to fall at the margin. IV verify that a related condition, sufficient for the small firm’s retail price to rise, is non-empty. They never check the consumer-harm condition itself. I check it. On IV’s both-binding Hotelling path, the consumer-harm condition is empty: it holds at no asymmetric equilibrium of the supplier’s problem. Proposition 3, which says the large buyer pays a lower wholesale price than the small rival, is unaffected.

1 Setup

The model is IV’s asymmetric-buyer setup specialized to the Hotelling downstream market they use for their consumer-side results. I keep their notation where possible and flag every departure. Readers with IV in hand can skim.¹

1.1 Downstream market

The economy has N independent local markets indexed by $n = 1, \dots, N$. Each market is a Hotelling duopoly. A unit mass of consumers is distributed uniformly

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1. Numbered equations in this section that have a character-for-character counterpart in IV 2011 carry a combined tag of the form “(k, IV j)”, where k is the local number and j is IV’s. Prose references to IV’s own conditions are written as “IV (10)” and “IV (12)” throughout to keep them distinct from this note’s own equation numbering.

on $[0, 1]$, two downstream firms locate at the endpoints, and consumers incur linear travel cost $t > 0$ per unit distance. Gross utility is the same at both firms and large enough that the market is fully covered. A consumer at location $x \in [0, 1]$ who buys from the firm at 0 pays $p_0 + tx$ out of pocket; from the firm at 1, $p_1 + t(1 - x)$. The indifferent consumer locates at $y = 1/2 + (p_1 - p_0)/(2t)$, and all consumers to the left of y buy from the firm at 0.

Each downstream firm incurs a per-unit processing cost $c \geq 0$ and buys an input from the upstream supplier at per-unit wholesale price w , so its gross marginal cost is $m := c + w$. Given gross costs (m_i, m_j) , the Bertrand–Nash equilibrium of the retail-pricing stage is

$$p_i^* = m_i + t + \frac{m_j - m_i}{3}, \quad y_i^* = \frac{1}{2} + \frac{m_j - m_i}{6t},$$

and the equilibrium profit of firm i per market is

$$\pi(m_i, m_j) = \frac{1}{2t} \left[t + \frac{m_j - m_i}{3} \right]^2. \quad (1)$$

The reduced-form profit π is decreasing in own marginal cost and increasing in rival marginal cost, so $\pi_1 < 0$ and $\pi_2 > 0$. IV's generic curvature condition is $\pi_{11} > 0$ together with the cross-partial sign condition $\pi_{12} < 0$ that drives the waterbed mechanism.

1.2 Upstream contracting and the supplier's problem

A single upstream supplier produces at constant marginal cost k and sells through per-unit wholesale contracts. The supplier makes simultaneous, publicly observable take-it-or-leave-it wholesale offers to each downstream firm. If a downstream firm rejects, it accesses an alternative source of supply at gross marginal cost $m := k + c$ after paying a fixed switching cost $F > 0$. The outside option is credible whenever F is low enough that the supplier cannot profitably raise the wholesale price past the point where rejection dominates acceptance.

One buyer—the *large buyer* L — owns $n_L \geq 2$ outlets, one in each of the first

n_L markets. In each of those n_L markets, L faces an independent *small rival* S . Markets $n > n_L$ play no role in what follows. Write w_I for the unique symmetric-buyer wholesale price from IV Proposition 1, w_L for the per-unit wholesale price the supplier charges L in every market where L operates, and w_S for the price charged to each S . Gross marginal costs are $m_L = c + w_L$ and $m_S = c + w_S$, so $m_i - m_j = w_i - w_j$.

The accept/reject decisions yield two participation constraints. Firm S accepts iff its profit under the contract beats its outside-option profit net of F :

$$\pi(m_S, m_L) \geq \pi(m, m_L) - F. \quad (2, \text{IV } 5)$$

The large buyer L evaluates its offer across all n_L outlets simultaneously. Rejecting triggers the fixed switching cost once, while the per-market profit differential is multiplied by n_L . The large buyer accepts iff

$$n_L \pi(m_L, m_S) \geq n_L \pi(m, m_S) - F. \quad (3, \text{IV } 5)$$

These are the asymmetric participation constraints; IV state them jointly as their equation (5). The division of F across n_L outlets on the L side is the formal source of the size premium. The large buyer's per-unit switching cost is F/n_L , which vanishes as $n_L \rightarrow \infty$.

IV, and this note, restrict attention to the low- F regime in which both participation constraints bind at the supplier's optimum, so (2) and (3) hold with equality:

$$\pi(m_S, m_L) - \pi(m, m_L) + F = 0, \quad (4, \text{IV } 20)$$

$$\pi(m_L, m_S) - \pi(m, m_S) + \frac{F}{n_L} = 0. \quad (5, \text{IV } 20)$$

The supplier chooses (w_S, w_L) with $w_S, w_L \geq k$ to maximize total profit across the n_L affected markets:

$$\max_{w_S, w_L \geq k} n_L \left[(w_S - k) y_S(m_S, m_L) + (w_L - k) y_L(m_L, m_S) \right] \quad \text{s.t. } (4), (5). \quad (6)$$

An *asymmetric equilibrium* is a solution (w_S^*, w_L^*) to (6) together with the implied Hotelling retail equilibrium in each affected market.

1.3 IV's asymmetric-buyer results engaged in this note

Three IV results are relevant.

Proposition (IV 2011, Proposition 3: wholesale-level waterbed). *In the asymmetric equilibrium of the both-binding regime, $w_L < w_I < w_S$. As n_L rises, w_L falls and w_S rises.*

This is the *wholesale-level* waterbed. Growing the large buyer widens the wholesale price spread, pushing w_L down and w_S up. It holds unconditionally in the both-binding regime and is untouched by this note.

Proposition (IV 2011, Proposition 5: small-firm retail-price waterbed). *Along the asymmetric-equilibrium path, a marginal decrease in w_L raises the small firm's retail price p_S when*

$$\frac{w_S - k}{3t} > y_S. \quad \text{IV (10)}$$

If the converse of IV (10) holds strictly, then all retail prices fall following a size-induced marginal reduction of the large buyer's wholesale price. IV further note that IV (10) can hold only if the large buyer controls sufficiently many outlets, competition is sufficiently intense (low t), and F is not too low.

Proposition 5 asks whether the waterbed raises *the small firm's* retail price alone. IV verify non-emptiness by exhibiting the closed-form limit family, in which IV (10) holds for $F/t \in (5/18, 3/8)$. This note does not challenge that verification.

Proposition (IV 2011, Proposition 6: consumer-surplus waterbed). *Along the asymmetric-equilibrium path, a marginal decrease in w_L lowers consumer surplus in the Hotelling model whenever*

$$2y_S \frac{2 - y_S}{1 + y_S} < \frac{w_S - k}{3t}. \quad \text{IV (12)}$$

Proposition 6 is the consumer-harm claim. It upgrades Proposition 5 from a statement about p_S alone to a statement about the marginal change in consumer surplus. IV condition (12) is strictly stronger than IV (10). The factor $2(2 - y_S)/(1 + y_S)$ exceeds 1 for all $y_S \in (0, 1/2]$. IV provide no non-emptiness argument for Proposition 6. I check it. IV condition (12) is vacuous on IV's both-binding Hotelling path. Section 2 rules it out directly from IV's $n_L \rightarrow \infty$ closed form. Section 3 rules it out at every finite- n_L KKT point of the supplier's problem. Corollary 1 upgrades that exclusion to interior local maximizers. Corollary 2 upgrades it to asymmetric equilibria.

IV condition (12) is an algebraic specialization of the sign of dCS/dw_L along the both-binding path. Since total demand is inelastic in Hotelling, consumer surplus in a single market is

$$CS = u - [y_S p_S + y_L p_L] - \frac{t}{2} [y_S^2 + y_L^2],$$

where u is the gross utility of consumption. The indifference condition $p_S + t y_S = p_L + t(1 - y_S)$ at the marginal consumer kills the dy_S/dw_L terms in dCS/dw_L , yielding

$$\frac{dCS}{dw_L} = - \left[y_S \frac{dp_S}{dw_L} + y_L \frac{dp_L}{dw_L} \right].$$

Along the both-binding path, IV's pass-through formula is

$$\frac{dw_S}{dw_L} = - \frac{w_S - k}{6t y_S}.$$

Using the Hotelling pricing equations,

$$\frac{dp_S}{dw_L} = \frac{1}{3} + \frac{2}{3} \frac{dw_S}{dw_L} = \frac{1}{3} - \frac{w_S - k}{9t y_S}, \quad \frac{dp_L}{dw_L} = \frac{2}{3} + \frac{1}{3} \frac{dw_S}{dw_L} = \frac{2}{3} - \frac{w_S - k}{18t y_S}.$$

2. IV state Proposition 6 as a sufficient condition. The paragraph below reproduces the Hotelling algebra from IV's equation (11) and shows that, along the both-binding path, IV (12) is in fact equivalent to $dCS/dw_L > 0$.

Substituting into the expression for dCS/dw_L and using $y_L = 1 - y_S$ gives

$$\frac{dCS}{dw_L} = -y_S \left[\frac{1}{3} - \frac{w_S - k}{9t y_S} \right] - (1 - y_S) \left[\frac{2}{3} - \frac{w_S - k}{18t y_S} \right] = -\frac{2 - y_S}{3} + \frac{(1 + y_S)(w_S - k)}{18t y_S}.$$

Since $t > 0$ and $y_S \in (0, 1)$ on the interior Hotelling path, the two sides are proportional with positive factor $(1 + y_S)/(6y_S)$, so

$$\frac{dCS}{dw_L} > 0 \iff \frac{w_S - k}{3t} > \frac{2y_S(2 - y_S)}{1 + y_S}.$$

Hence IV condition (12) is exactly the sign condition for $dCS/dw_L > 0$ along the both-binding Hotelling path.

1.4 Reduced variables and the waterbed regime

Throughout the proofs I work in reduced-form variables

$$\tilde{\zeta} := \frac{w_S - k}{3t}, \quad \eta := \frac{w_L - k}{3t},$$

so $\tilde{\zeta}$ and η are the small-firm and large-buyer supplier markups normalized by $3t$. The Hotelling share formula, combined with $w_S - w_L = 3t(\tilde{\zeta} - \eta)$, gives

$$y_S = \frac{1 - \tilde{\zeta} + \eta}{2}, \quad y_L = \frac{1 + \tilde{\zeta} - \eta}{2}, \quad \tilde{\zeta} - \eta = 1 - 2y_S.$$

In these variables, IV's two conditions become

$$\text{IV (10): } \tilde{\zeta} > y_S, \quad \text{IV (12): } \tilde{\zeta} > \frac{2y_S(2 - y_S)}{1 + y_S}.$$

The *waterbed regime* is the subset of parameters on which IV Proposition 3 delivers $w_L < w_I < w_S$. In reduced form, this is $\tilde{\zeta} > \eta \geq 0$. Here $\tilde{\zeta} > \eta$ is equivalent to $y_S < 1/2$, with $y_S = 1/2$ the symmetric benchmark, while $\eta \geq 0$ is the separate lower-bound restriction $w_L \geq k$. The proofs below work in the *strict waterbed regime* $\tilde{\zeta} > \eta$, equivalently $y_S < 1/2$.

1.5 The large-buyer limit

The short proof of Section 2 borrows one characterization from IV: the $n_L \rightarrow \infty$ limit of the reduced binding constraints. IV derive it in the appendix to their Proposition 5; it also falls out of the reduced binding constraints directly.

Lemma 1 (IV 2011, Proposition 5 limit characterization). *In the $n_L \rightarrow \infty$ limit of the both-binding regime, the large buyer's wholesale price satisfies $w_L \rightarrow k$ (equivalently $\eta \rightarrow 0$), and the small firm's reduced markup $\xi = (w_S - k)/(3t)$ satisfies*

$$\xi(2 - \xi) = \frac{2F}{t}.$$

The unique feasible solution has $\xi \in (0, 1/2)$ for all $F/t \in (0, 3/8)$, and $\xi = 1/2$ at $F/t = 3/8$.

Proof. Substituting the Hotelling π from (1) and the reduced variables into the binding constraints (4), (5) yields the algebraic equivalents

$$2\xi + 2\xi\eta - \xi^2 = \frac{2F}{t}, \tag{7}$$

$$2\eta + 2\xi\eta - \eta^2 = \frac{2F}{t n_L}. \tag{8}$$

As $n_L \rightarrow \infty$, the right-hand side of (8) vanishes, forcing $\eta(2 + 2\xi - \eta) = 0$. Feasibility $y_L = (1 + \xi - \eta)/2 \geq 0$ gives $\eta \leq 1 + \xi < 2 + 2\xi$, so $2 + 2\xi - \eta > 0$ and $\eta = 0$. Substituting $\eta = 0$ into (7) gives $\xi(2 - \xi) = 2F/t$, equivalently $g(\xi) := \xi^2 - 2\xi + 2F/t = 0$.

For $F/t < 3/8$, three sign checks: $g(0) = 2F/t > 0$; $g(1/2) = 2F/t - 3/4 < 0$; $g(\xi) \rightarrow +\infty$ as $\xi \rightarrow +\infty$. Since g is a convex parabola with strict sign changes on $(0, 1/2)$ and $(1/2, \infty)$, it has exactly one root in each. Feasibility $y_S = (1 - \xi)/2 > 0$ forces $\xi < 1$, and $g(1) = 2F/t - 1 < 0$ for $F/t < 1/2$, so the upper root exceeds 1 and the feasible root lies in $(0, 1/2)$ uniquely. At $F/t = 3/8$, g factors as $g(\xi) = (\xi - 1/2)(\xi - 3/2)$, so the roots are $1/2$ and $3/2$; feasibility $\xi < 1$ selects $\xi = 1/2$. ■

This is the only ingredient Section 2 takes from IV.

2 A short proof via IV's own large-buyer limit

Proposition 1 (Proposition 6 has empty support in the $n_L \rightarrow \infty$ limit). *In the explicit equilibrium family IV construct at $n_L \rightarrow \infty$, IV condition (12) never holds. Proposition 6 has empty support in the regime where IV verify that Proposition 5 is non-empty.*

Proof. By Lemma 1, in the $n_L \rightarrow \infty$ limit of the both-binding regime, $\eta = 0$ and $\bar{\zeta} \leq 1/2$ for $F/t \in (0, 3/8]$, with $\bar{\zeta} < 1/2$ strict on the interior $F/t \in (0, 3/8)$.

With $\eta = 0$, $\bar{\zeta} = 1 - 2y_S$, equivalently $y_S = (1 - \bar{\zeta})/2$. Substitute into IV condition (12):

$$\bar{\zeta} > \frac{2y_S(2 - y_S)}{1 + y_S} = \frac{(1 - \bar{\zeta})(3 + \bar{\zeta})/2}{(3 - \bar{\zeta})/2} = \frac{(1 - \bar{\zeta})(3 + \bar{\zeta})}{3 - \bar{\zeta}}.$$

Multiplying through by the strictly positive $3 - \bar{\zeta}$,

$$\bar{\zeta}(3 - \bar{\zeta}) > (1 - \bar{\zeta})(3 + \bar{\zeta}).$$

Expanding both sides,

$$3\bar{\zeta} - \bar{\zeta}^2 > 3 - 2\bar{\zeta} - \bar{\zeta}^2,$$

so $5\bar{\zeta} > 3$, i.e.,

$$\bar{\zeta} > \frac{3}{5}.$$

This contradicts $\bar{\zeta} \leq 1/2$ from Lemma 1. IV condition (12) cannot hold anywhere in the equilibrium family for which IV establish that both participation constraints bind. ■

In the $n_L \rightarrow \infty$ limit, the large buyer's participation constraint slackens so completely that the supplier's only remaining problem is to extract rent from the small firm. That problem caps the wholesale markup at $\bar{\zeta} < 1/2$. IV condition (12), which is what IV need for consumer surplus to fall at the margin, requires $\bar{\zeta} > 3/5$. The waterbed is too weak to clear the bar IV's own condition (12) sets.

3 Finite buyer size: the consumer-harm condition is vacuous

Section 2 showed that IV's own verification strategy fails for Proposition 6: the limit construction they use to check Proposition 5 refutes Proposition 6. That leaves open whether some other equilibrium of the Hotelling model, at finite n_L , might satisfy condition (12). It does not. IV condition (12) fails at every KKT point of the supplier's problem in the strict waterbed regime, for any $F > 0$ and any $n_L \geq 2$. The argument works directly from the KKT conditions and exhausts the possible active sets of the participation constraints.

Normalize the supplier's profit per affected market by $3t$:

$$\Phi(\xi, \eta) := \xi y_S + \eta y_L = \frac{\xi + \eta}{2} + \xi \eta - \frac{\xi^2 + \eta^2}{2},$$

where $y_L = 1 - y_S = (1 + \xi - \eta)/2$. The participation constraints (4)–(5), in slack form, become

$$g_S(\xi, \eta) := \frac{F}{t} - \xi - \xi \eta + \frac{\xi^2}{2} \geq 0, \quad (9)$$

$$g_L(\xi, \eta) := \frac{F}{tn_L} - \eta - \xi \eta + \frac{\eta^2}{2} \geq 0, \quad (10)$$

where $g_i \geq 0$ iff PC_i is satisfied. (Equivalently, PC_S binds iff $2\xi\eta + 2\xi - \xi^2 = 2F/t$, the Hotelling specialization of IV's equation (23). At $\xi = 0$, $g_S = F/t > 0$, confirming that the slack direction is correct.) The supplier maximizes Φ subject to $g_S \geq 0$, $g_L \geq 0$, $\xi, \eta \geq 0$. The Lagrangian, with multipliers on the non-negativity constraints included explicitly, is

$$\mathcal{L} = \Phi + \mu_S g_S + \mu_L g_L + \alpha \xi + \beta \eta,$$

with $\mu_S, \mu_L, \alpha, \beta \geq 0$.

Theorem 1 (Vacuity of IV condition (12) at KKT points). *For any $F > 0$ and any $n_L \geq 2$, no KKT point of the supplier's Hotelling problem in the strict waterbed regime*

$\bar{\xi} > \eta \geq 0, 0 < y_S < 1$ satisfies IV condition (12).

Proof sketch (full proof in the [Appendix](#)). The strict waterbed regime forces $\bar{\xi} > 0$, so complementary slackness gives $\alpha = 0$. The boundary $\eta = 0$ is ruled out: there, $g_L = F/(tn_L) > 0$ forces $\mu_L = 0$, and the remaining stationarity conditions give the multiplier on the $\eta \geq 0$ constraint as $\beta = -1/(2(1 - \bar{\xi})) < 0$, contradicting $\beta \geq 0$. So every KKT point has $\eta > 0$.

At interior KKT points, we exhaust the four active-set configurations of the participation constraints g_S, g_L . Cases 1 and 2 yield no KKT points in the strict waterbed regime (Case 1 by $1 = 0$; Case 2 because the stationarity-implied candidate violates $g_L \geq 0$). Cases 3 and 4 produce KKT points, but the stationarity conditions force $\bar{\xi} \leq 1/(3 - 4y_S)$. IV condition (12) requires the strictly larger bound $\bar{\xi} > 2y_S(2 - y_S)/(1 + y_S)$. Combining these reduces to the cubic

$$P(y_S) := (2y_S - 1)(4y_S^2 - 9y_S + 1) > 0 \quad \text{on } (1/4, 1/2),$$

which holds because both factors are strictly negative there. Contradiction in every case. ■

Corollary 1 (Vacuity at interior local maximizers). *Under the same hypotheses, IV condition (12) is vacuously false at every interior local maximizer of the supplier's Hotelling problem in the strict waterbed regime.*

Proof. The corollary adds no new algebra. It converts the theorem's KKT exclusion into an exclusion of interior local maximizers via the Mangasarian–Fromovitz constraint qualification. Let $(\bar{\xi}, \eta)$ be any interior feasible point in the theorem's regime, so $\bar{\xi} > 0$ and $\eta > 0$. The non-negativity constraints are slack, so the only possibly active inequalities are g_S and g_L , with gradients

$$\nabla g_S = (-2y_S, -\bar{\xi}), \quad \nabla g_L = (-\eta, -2y_L).$$

Take the common slackening direction $d := (-1, -1)$. Then

$$\nabla g_S \cdot d = 2y_S + \bar{\xi} > 0, \quad \nabla g_L \cdot d = \eta + 2y_L > 0,$$

because $y_S, y_L, \xi, \eta > 0$ in the strict waterbed regime. The Mangasarian–Fromovitz constraint qualification therefore holds at every interior feasible point in the theorem’s domain. Every interior local maximizer is therefore a KKT point (Bertsekas 1999, Proposition 3.3.8), and Theorem 1 rules all such points out. ■

Corollary 2 (Vacuity at asymmetric equilibria). *Under the same hypotheses, IV condition (12) is vacuously false at every asymmetric equilibrium of the supplier’s Hotelling problem in the strict waterbed regime.*

Proof. An asymmetric equilibrium is, by definition, a solution to (6), hence a global maximizer of the supplier’s objective over the feasible set and therefore a local maximizer. It remains to show that such a maximizer is interior.

In the strict waterbed regime, $\xi > \eta \geq 0$ already gives $\xi > 0$, so the only non-negativity boundary is $\eta = 0$. Suppose $(\xi, 0)$ were feasible in the strict waterbed regime. Then $y_S = (1 - \xi)/2$, so $0 < \xi < 1$, and $g_L = F/(tn_L) > 0$. Consider the direction

$$d := \left(-\frac{1}{1-\xi}, 1\right).$$

At $\eta = 0$, the first-order change in the small firm’s participation slack is

$$\nabla g_S \cdot d = (\xi - 1) \left(-\frac{1}{1-\xi}\right) - \xi = 1 - \xi > 0.$$

Thus if $g_S = 0$ at $(\xi, 0)$, moving a small amount in direction d strictly relaxes that constraint; if $g_S > 0$, feasibility persists by continuity. Since $g_L > 0$ at $(\xi, 0)$, the large buyer’s participation constraint also remains feasible for all sufficiently small steps in the same direction. Finally, the directional derivative of the supplier’s objective is

$$\nabla \Phi \cdot d = \left(\frac{1}{2} - \xi\right) \left(-\frac{1}{1-\xi}\right) + \left(\frac{1}{2} + \xi\right) = \frac{\xi(3-2\xi)}{2(1-\xi)} > 0,$$

because $0 < \xi < 1$. So every sufficiently small feasible move in direction d raises the supplier’s objective. Therefore no feasible point with $\eta = 0$ can be a local maximizer.

Every asymmetric equilibrium in the theorem's regime is therefore an interior local maximizer. Corollary 1 applies and rules out IV condition (12) at every such equilibrium. ■

Proposition 6 is not false. IV state it as an if-then, and the implication is valid. But IV (12) never holds at the asymmetric equilibrium points IV analyze. The antecedent has empty support, so the proposition is vacuously true. It delivers no consumer-harm conclusion.

The economic reason runs through the small firm's participation constraint. The waterbed works through an endogenous outside option. When the supplier lowers w_L to secure the large buyer's participation, the large firm prices more aggressively and steals market share from the small firm. That reduces the small firm's outside-option value, slackening its participation constraint and letting the supplier raise w_S . That is the waterbed.

But the small firm accepts a higher w_S only up to the point where the deal still beats walking away. That threshold pins w_S down. In the Hotelling specialization it pins w_S below $1/(3 - 4y_S)$ in reduced variables. The waterbed can push the small firm to accept a higher wholesale price than it would in the symmetric benchmark, but only as far as the small firm's revised outside option permits.

Proposition 5 survives. It asks whether the PC shift raises the small firm's retail price p_S alone. That is a strictly weaker test than Proposition 6, which requires the shift to raise the share-weighted average of p_S and p_L . IV's non-emptiness witness for Proposition 5 lives in the $n_L \rightarrow \infty$ limit and satisfies IV (10) but, by Section 2, falls strictly below the IV (12) threshold. The analogous finite- n_L question for Proposition 5 is open.

The conclusion is confined to IV's Hotelling specialization. Their general analysis with unspecified π is unaffected. Proposition 3, the wholesale-level waterbed $w_L < w_I < w_S$, holds unconditionally in the both-binding regime. This note rules out the consumer-harm conclusion at asymmetric equilibria of the supplier's Hotelling problem in the strict waterbed regime. Whether some other downstream specification admits a non-empty consumer-harm region remains open.

Appendix Full proof of Theorem 1

The strict waterbed regime $\bar{\zeta} > \eta \geq 0$ forces $\bar{\zeta} > 0$, so complementary slackness on $\bar{\zeta} \geq 0$ gives $\alpha = 0$ throughout. We first dispose of the boundary case $\eta = 0$, then handle the interior case $\eta > 0$ via the four active sets of g_S and g_L .

Boundary case $\eta = 0$. At $\eta = 0$ we have $g_L = F/(tn_L) > 0$, so $\mu_L = 0$ by complementary slackness, and $y_S = (1 - \bar{\zeta})/2$, so $2y_S = 1 - \bar{\zeta}$. The $\bar{\zeta}$ -stationarity equation $(2y_S - \frac{1}{2}) - 2y_S \mu_S = 0$ gives

$$\mu_S = \frac{2y_S - 1/2}{2y_S} = \frac{1/2 - \bar{\zeta}}{1 - \bar{\zeta}}.$$

The η -stationarity equation reads $(\frac{3}{2} - 2y_S) - \bar{\zeta} \mu_S + \beta = 0$, i.e., $\beta = \bar{\zeta} \mu_S - (1/2 + \bar{\zeta})$. Substituting μ_S and combining over the common denominator $1 - \bar{\zeta}$,

$$\beta = \frac{\bar{\zeta}(1/2 - \bar{\zeta}) - (1/2 + \bar{\zeta})(1 - \bar{\zeta})}{1 - \bar{\zeta}}.$$

Expand the two products: $\bar{\zeta}(1/2 - \bar{\zeta}) = \bar{\zeta}/2 - \bar{\zeta}^2$ and $(1/2 + \bar{\zeta})(1 - \bar{\zeta}) = 1/2 + \bar{\zeta}/2 - \bar{\zeta}^2$. The $\bar{\zeta}/2$ and $\bar{\zeta}^2$ terms cancel exactly, leaving

$$\beta = \frac{-1/2}{1 - \bar{\zeta}} = -\frac{1}{2(1 - \bar{\zeta})}.$$

The strict waterbed regime requires $0 < y_S$, i.e., $\bar{\zeta} < 1$, so $\beta < 0$. This contradicts $\beta \geq 0$. Hence $\eta > 0$ at every KKT point in the strict waterbed regime, and complementary slackness gives $\beta = 0$ as well.

Interior case $\eta > 0$. With $\alpha = \beta = 0$, the stationarity conditions $\partial_{\bar{\zeta}} \mathcal{L} = 0$ and $\partial_{\eta} \mathcal{L} = 0$, which we label $(S_{\bar{\zeta}})$ and (S_{η}) , are

$$\begin{aligned} (2y_S - \frac{1}{2}) - 2y_S \mu_S - \eta \mu_L &= 0, \\ (\frac{3}{2} - 2y_S) - \bar{\zeta} \mu_S - 2y_L \mu_L &= 0, \end{aligned}$$

using $1 + \eta - \bar{\zeta} = 2y_S$ and $1 + \bar{\zeta} - \eta = 2y_L$. In the strict waterbed regime $y_S < 1/2$, so $\partial_{\eta} \Phi = \frac{3}{2} - 2y_S > \frac{1}{2} > 0$: the supplier's profit strictly increases in η , the large

buyer's wholesale markup. Direct expansion gives the algebraic identity, which we label (+),

$$g_S - g_L = (\bar{\zeta} - \eta) \left(\frac{\bar{\zeta} + \eta}{2} - 1 \right) + \frac{F}{t} \left(1 - \frac{1}{n_L} \right),$$

which we will use repeatedly. We exhaust the four possible active sets for g_S, g_L .

Case 1: both PCs slack. Complementary slackness gives $\mu_S = \mu_L = 0$, and $(S_{\bar{\zeta}}), (S_{\eta})$ collapse to $\nabla\Phi = 0$: $\frac{1}{2} + \eta - \bar{\zeta} = 0$ and $\frac{1}{2} + \bar{\zeta} - \eta = 0$. Adding gives $1 = 0$. Contradiction.

Case 2: only PC_S binds. Then $\mu_L = 0$. Solving $(S_{\bar{\zeta}})$ and (S_{η}) for μ_S and equating:

$$\mu_S = \frac{4y_S - 1}{4y_S} = \frac{3 - 4y_S}{2\bar{\zeta}}.$$

The first form forces $y_S \geq 1/4$, hence $y_S \in (1/4, 1/2)$ in the strict waterbed regime. Equating and solving (with $\bar{\zeta} - \eta = 1 - 2y_S$):

$$\bar{\zeta} = \frac{2y_S(3 - 4y_S)}{4y_S - 1}, \quad \eta = \frac{1}{4y_S - 1}.$$

We claim this candidate is infeasible because $g_L < 0$ at it. From the constraint-difference identity (+) with $g_S = 0$,

$$g_L = -(\bar{\zeta} - \eta) \left(\frac{\bar{\zeta} + \eta}{2} - 1 \right) - \frac{F}{t} \left(1 - \frac{1}{n_L} \right).$$

The second term is strictly negative ($F > 0, n_L > 1$). It suffices to show $\bar{\zeta} + \eta > 2$ at the candidate, since $\bar{\zeta} > \eta$ then makes the first term strictly negative as well. Direct computation:

$$\frac{\bar{\zeta} + \eta}{2} = \frac{2y_S(3 - 4y_S) + 1}{2(4y_S - 1)} = \frac{-8y_S^2 + 6y_S + 1}{2(4y_S - 1)}.$$

The inequality $\frac{\bar{\zeta} + \eta}{2} > 1$ becomes $-8y_S^2 + 6y_S + 1 > 2(4y_S - 1)$, equivalently $8y_S^2 + 2y_S - 3 < 0$. This factors as $(2y_S - 1)(4y_S + 3) < 0$, which holds strictly on $(1/4, 1/2)$. So $\bar{\zeta} + \eta > 2$, hence $g_L < 0$. The Case 2 candidate is infeasible: no Case 2 KKT point exists.

Case 3: only PC_L binds. Then $\mu_S = 0$. Solving (S_{ξ}) and (S_{η}) for μ_L and equating:

$$\mu_L = \frac{4y_S - 1}{2\eta} = \frac{3 - 4y_S}{2(1 + \xi - \eta)}.$$

The first form forces $y_S \geq 1/4$, hence $y_S \in (1/4, 1/2)$. Solving (with $\xi - \eta = 1 - 2y_S$):

$$\xi = \frac{1}{3 - 4y_S}.$$

Combine with IV condition (12), $\xi(1 + y_S) > 2y_S(2 - y_S)$:

$$\frac{1 + y_S}{3 - 4y_S} > 2y_S(2 - y_S) \iff 1 + y_S > 2y_S(2 - y_S)(3 - 4y_S),$$

since $3 - 4y_S > 0$. Expanding the RHS yields the cubic

$$P(y_S) := 8y_S^3 - 22y_S^2 + 11y_S - 1 < 0.$$

Since $P(1/2) = 0$, factor: $P(y_S) = (2y_S - 1)(4y_S^2 - 9y_S + 1)$. The quadratic factor has roots $(9 \pm \sqrt{65})/8 \approx 0.117$ and 2.133 , both outside $(1/4, 1/2)$, and is strictly negative at both endpoints: $4(1/4)^2 - 9(1/4) + 1 = -1$ and $4(1/2)^2 - 9(1/2) + 1 = -5/2$. So it is strictly negative on the entire interval. The linear factor $(2y_S - 1)$ is also strictly negative on $(1/4, 1/2)$. Both factors strictly negative, so $P(y_S) > 0$ on $(1/4, 1/2)$. IV condition (12) requires $P(y_S) < 0$. Contradiction.

Case 4: both PCs bind. Then $g_S = g_L = 0$. The stationarity system $(S_{\xi}), (S_{\eta})$ is linear in (μ_S, μ_L) with matrix

$$M = \begin{pmatrix} 2y_S & \eta \\ \xi & 2y_L \end{pmatrix}, \quad D := \det M = 4y_S y_L - \xi \eta.$$

The same system admits a determinant-free elimination. Multiply the rearranged stationarity equations

$$2y_S \mu_S + \eta \mu_L = 2y_S - \frac{1}{2}, \quad \xi \mu_S + 2y_L \mu_L = \frac{3}{2} - 2y_S$$

by $2y_L$ and $-\eta$, respectively, and add. The μ_L -terms cancel, giving

$$\mu_S \cdot D = (2y_S - \frac{1}{2})(2y_L) - \eta(\frac{3}{2} - 2y_S).$$

Using $2y_S = 1 + \eta - \zeta$ and $2y_L = 1 + \zeta - \eta$, this becomes

$$\mu_S \cdot D = (\frac{1}{2} + \eta - \zeta)(1 + \zeta - \eta) - \eta(\frac{1}{2} + \zeta - \eta).$$

Expanding the first product yields $\frac{1}{2} - \frac{\zeta}{2} + \frac{\eta}{2} + 2\zeta\eta - \zeta^2 - \eta^2$; subtracting $\eta(\frac{1}{2} + \zeta - \eta) = \frac{\eta}{2} + \zeta\eta - \eta^2$ leaves the S -multiplier identity (\dagger_S),

$$\mu_S \cdot D = \frac{1}{2} - \frac{\zeta}{2} + \zeta\eta - \zeta^2 = \frac{1}{2}[1 - \zeta(1 + 2\zeta - 2\eta)] = \frac{1}{2}[1 - \zeta(3 - 4y_S)],$$

using $1 + 2\zeta - 2\eta = 1 + 2(1 - 2y_S) = 3 - 4y_S$. By the symmetry $\zeta \leftrightarrow \eta$, $y_S \leftrightarrow y_L$, which sends $3 - 4y_S$ to $4y_S - 1$, or equivalently by multiplying the first rearranged stationarity equation by $-\zeta$ and the second by $2y_S$ and adding, we obtain the companion L -multiplier identity (\dagger_L),

$$\mu_L \cdot D = \frac{1}{2}[1 - \eta(4y_S - 1)].$$

These three identities have distinct roles in what follows. The constraint-difference identity (\dagger) controls $\zeta + \eta$ in the binding regime, and the multiplier identities (\dagger_S) and (\dagger_L) convert the multiplier sign conditions $\mu_S, \mu_L \geq 0$ into bounds on ζ and η .

Claim. At every Case 4 KKT point in the strict waterbed regime, $D > 0$.

Proof of claim. Suppose for contradiction $D \leq 0$. From the constraint-difference identity (\dagger) with $g_S = g_L = 0$,

$$(\zeta - \eta)\left(\frac{\zeta + \eta}{2} - 1\right) = -\frac{F}{t}\left(1 - \frac{1}{n_L}\right) < 0,$$

since $F > 0$ and $n_L > 1$. With $\zeta > \eta$, this forces

$$\zeta + \eta < 2 \quad (*).$$

From (\ddagger_L) and $\mu_L \geq 0, D \leq 0$, we get $1 - \eta(4y_S - 1) \leq 0$, i.e., $\eta(4y_S - 1) \geq 1$. With $\eta \geq 0$, this requires $4y_S - 1 > 0$, i.e., $y_S > 1/4$, and $\eta \geq 1/(4y_S - 1)$. Symmetrically, (\ddagger_S) gives $\xi \geq 1/(3 - 4y_S)$.

Adding the two lower bounds:

$$\xi + \eta \geq \frac{1}{3 - 4y_S} + \frac{1}{4y_S - 1} = \frac{2}{(3 - 4y_S)(4y_S - 1)}.$$

The right-hand side exceeds 2 iff $(3 - 4y_S)(4y_S - 1) < 1$, equivalently $16y_S^2 - 16y_S + 4 > 0$, i.e., $4(2y_S - 1)^2 > 0$. This holds strictly for $y_S \neq 1/2$, in particular on $(1/4, 1/2)$. So $\xi + \eta > 2$, contradicting (*). Hence $D > 0$. ■

By the claim, $D > 0$, so (\ddagger_S) gives $\mu_S \geq 0 \iff \xi \leq 1/(3 - 4y_S)$. Combined with $\eta \geq 0 \iff \xi \geq 1 - 2y_S$:

$$1 - 2y_S \leq \xi \leq \frac{1}{3 - 4y_S}.$$

Non-emptiness requires $(1 - 2y_S)(3 - 4y_S) \leq 1$, equivalently $(4y_S - 1)(y_S - 1) \leq 0$, i.e., $y_S \in [1/4, 1]$. With $y_S < 1/2$ (waterbed): $y_S \in [1/4, 1/2)$. (At $y_S = 1/4$ the bounds force $\eta = 0$, which contradicts $g_L = 0$, so $y_S > 1/4$ strictly.)

Combine IV condition (12) with the upper bound $\xi \leq 1/(3 - 4y_S)$:

$$\frac{2y_S(2 - y_S)}{1 + y_S} < \xi \leq \frac{1}{3 - 4y_S} \implies 2y_S(2 - y_S)(3 - 4y_S) < 1 + y_S \iff P(y_S) < 0,$$

where P is the cubic from Case 3. The same factoring gives $P(y_S) > 0$ on $(1/4, 1/2)$. Contradiction.

Every case is ruled out. No KKT point of the supplier's problem in the strict waterbed regime satisfies IV condition (12). ■

References

Bertsekas, Dimitri P. 1999. *Nonlinear Programming*. 2nd ed. Belmont, MA: Athena Scientific.

Inderst, Roman, and Tommaso M. Valletti. 2011. "Buyer Power and the "Waterbed Effect"." *Journal of Industrial Economics* 59 (1): 1–20.